First-principles formulation of resonance broadened quasilinear theory near an instability threshold

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Weekly PPPL Energetic Particle Virtual Group Meeting, April 28th 2020

Quasilinear theory is a reduced approach to kinetic instabilities

In a regime where orbits are stochastic (no effective particle trapping in resonances), the kinetic (Vlasov) description of phase mixing can be approximated by an irreversible, diffusive process

$$f\left(x,v,t\right) \qquad\qquad f\left(v,t\right)=\left\langle f\left(x,v,t\right)\right\rangle$$

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{F}{m}\frac{\partial f}{\partial v}=C[f] \longrightarrow \frac{\partial f}{\partial t}+\frac{\partial}{\partial v}D\frac{\partial f}{\partial v}=C[f]$$

For quasilinear theory to be valid, the linear mode properties (e.g., eigenstructure and resonance condition) should not change in time

Quasilinear diffusion theory was independently proposed by

A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Sov. Phys. Usp. 4, 332 (1961).

W. Drummond and D. Pines, Nucl. Fusion Suppl. 2(Pt. 3), 1049 (1962).

Later generalized to action-angle variables:

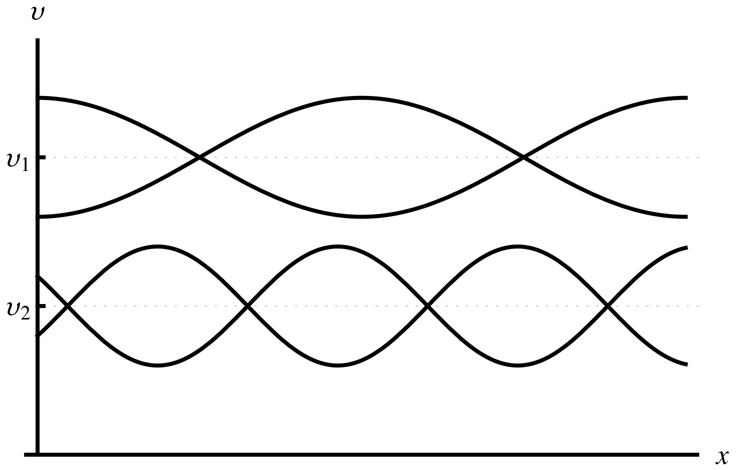
A. N. Kaufman, Phys. Fluids 15, 1063 (1972).

Historically, resonance overlap (Chirikov criterion) has been invoked to justify the applicability of QL theory

$$v_{\mathrm{tr}1} + v_{\mathrm{tr}2} \gtrsim \left| v_{\mathrm{p}2} - v_{\mathrm{p}1} \right|$$

In this case, most trapped particles will not “belong” to a particular wave anymore but will be “shared” by the two waves.

- Intrinsic stochastic diffusion: due to interaction with broad spectrum
- Extrinsic stochasticity: by collisions inducing randomization of phase



I. Y. Dodin, Lectures notes on Waves in Plasmas, Princeton University

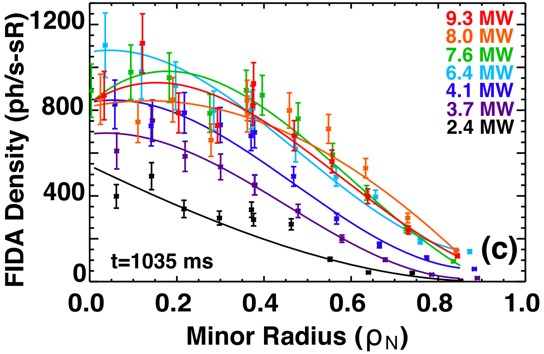
The end goal of this talk is to show that in the presence of collisions, a QL theory can be formulated from first principles near marginal stability, even for a single resonance. Interesting properties emerge:

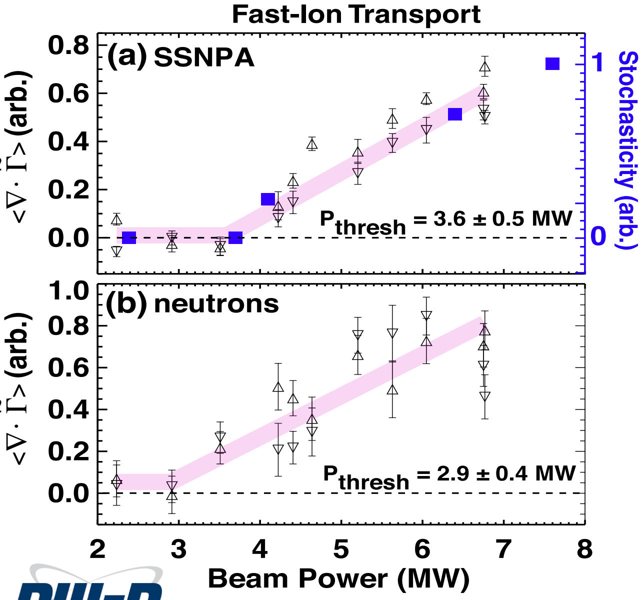
(i) it recovers the saturation level predicted by nonlinear theory
(ii) the resonance function can be analytically calculated

Critical gradient behavior in DIII-D suggests that quasilinear modeling is a viable modeling tool for fast ion relaxation

DIII-D critical gradient experiments

- stiff, resilient fast ion profiles as beam power varies
-stochastic fast ion transport (mediated by overlapping resonances) gives credence in using a quasilinear approach





- Fully nonlinear modeling of fast ion interaction with Alfvénic modes in a realistic tokamak is numerically expensive
- Simulations need to cope with the simultaneous excitation of multiple unstable Alfvénic instabilities

C. Collins et al, PRL 2016

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Early development of broadened quasilinear theory

- The broadening of resonances is a ubiquitous phenomenon in physics (e.g., in atomic spectra)
- In plasma physics, broadened strong turbulence theories for dense spectra have been developed (e.g., Dupree, Phys. Fluids 1966);

For beam-plasma interaction in a tokamak, consider canonical variables of actions J and angles φ. In a tokamak, J is a combination of $(\mathcal{E}, P_\varphi, \mu)$

$$\dot{\varphi} = \partial H_0 (J) / \partial J \equiv \Omega (J)$$

The line broadening model ($\delta(\Omega) \to \mathcal{R}(\Omega)$):

$$\frac{\partial f(\Omega,t)}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial \Omega}\left[\left|\omega_b^2\right|^2 \mathcal{R}(\Omega)\frac{\partial f(\Omega,t)}{\partial \Omega}\right] = C[f, F_0]$$

$$d\left|\omega_b^2\right|^2/dt \;=\; 2\left(\gamma_L(t) - \gamma_d\right)\left|\omega_b^2\right|^2$$

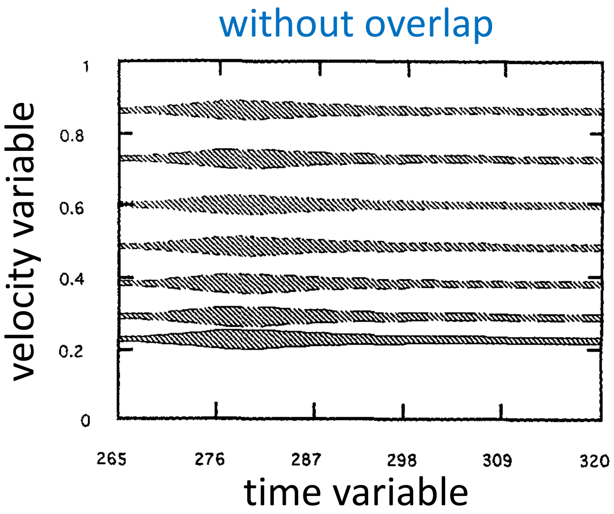
$$\gamma_L(t) \;=\; \tfrac{\pi}{4}\int_{-\infty}^{\infty} d\Omega \mathcal{R}\tfrac{\partial f(\Omega,t)}{\partial \Omega}$$

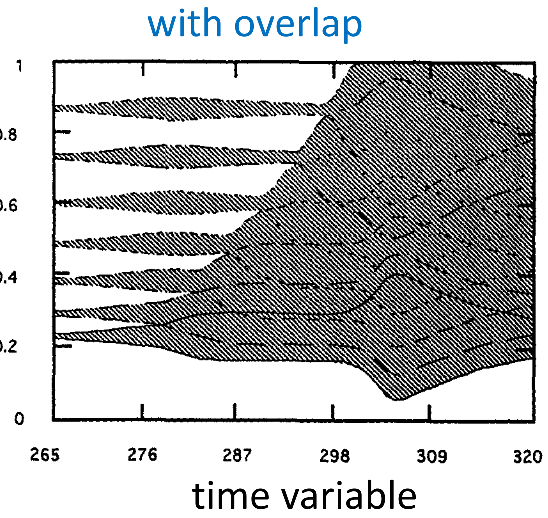
- $\mathcal{R}$ is an arbitrary resonance function (usually taken as in flat-top form) with $\int_{-\infty}^{\infty}\mathcal{R}(\Omega)d\Omega \;=\; 1$
- ω_b is the trapping (bounce) frequency at the elliptic point (proportional to square root of mode amplitude)

H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).

The overlapping of resonances lead to losses due to global diffusion

- Designed to address both regimes of isolated and overlapping resonances
  - the fast ion distribution function relaxes while self-consistently evolving the amplitude of modes





H. Berk, B. Breizman, J. Fitzpatrick, and H. Wong, Nucl. Fusion 35, 1661 (1995).

Determining the parametric dependencies of the broadening from single mode saturation levels

The broadening is assumed with the parametric form $\Delta\Omega = a\omega_b + b\nu_{eff}$ where the coefficients a and b are determined in order to enforce QL theory to replicate known nonlinear saturation levels:

Limit near marginal stability[3] $\omega_b = 1.18\nu_{\mathrm{eff}} \left(\frac{\gamma_{L0}-\gamma_d}{\gamma_{L0}}\right)^{1/4}$

→ $b = 3.1$

Limit far from marginal stability[4] $\omega_b = 1.2\nu_{\mathrm{eff}} \left(\frac{\gamma_{L0}-\gamma_d}{\gamma_d}\right)^{1/3}$

→ $a = 2.7$

Resonance-broadened quasilinear formalism can cope with both situations of isolated and overlapping modes

[3]H. L. Berk et al. Plasma Phys. Rep, 23(9), 1997 [4]H. L. Berk and B. N. Breizman. Phys. Fluids B, 2(9), 1990

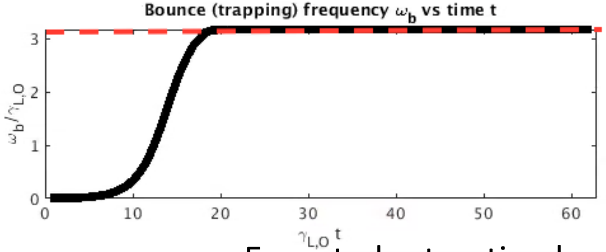
Broadening was adjusted to replicate analytical predictions for the mode saturation amplitude of single modes

Definitions: initial linear growth rate γ_L, mode damping rate γ_d and trapping (bounce) frequency ω_b (proportional to square root of mode amplitude)

Collisionless case

- Undamped case

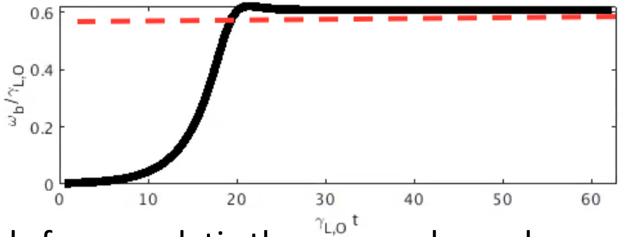
$$\omega_b \cong 3.2\gamma_L$$



Collisional cases

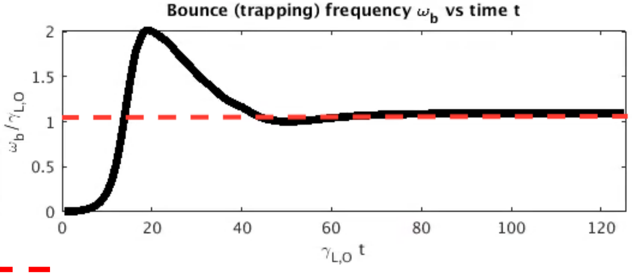
- Close to marginal stability: $\nu_{\text{eff}} \gg \omega_b$

$$\omega_b = 1.18\nu_{\text{eff}}\left(\frac{\gamma_L - \gamma_d}{\gamma_d}\right)^{1/4}$$



- Far from marginal stability: $\omega_b \gg \nu_{\text{eff}}$

$$\omega_b = 1.2\nu_{\text{eff}}\left(\frac{\gamma_L - \gamma_d}{\gamma_d}\right)^{1/3}$$



Expected saturation levels from analytic theory are shown by – – –

First-principle analytical determination of the collisional resonance broadening – part I

Start with the kinetic equation: $\frac{\partial f}{\partial t}+\Omega\frac{\partial f}{\partial\varphi}+Re\left(\omega_b^2 e^{i\varphi}\right)\frac{\partial f}{\partial\Omega}=C\left[f,F_0\right]$

$$C\left[f,F_0\right]=\begin{cases}\nu_K\left(F_0-f\right)\\ \nu_{scatt}^3\partial^2\left(f-F_0\right)/\partial\Omega^2\\ \text{(from collisions, turbulence,...)}\end{cases}$$

Periodicity over the canonical angle allows the distribution to be written as a Fourier series:

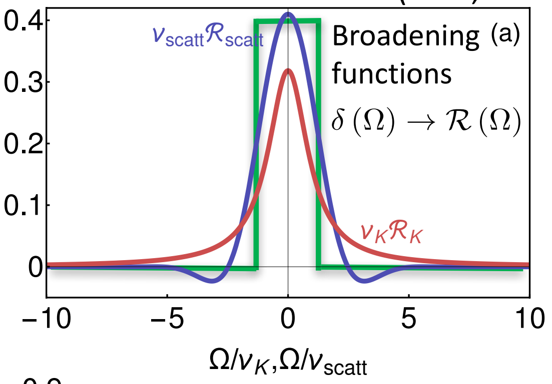
$$f\left(\varphi,\Omega,t\right)=F_0\left(\Omega\right)+f_0\left(\Omega,t\right)+\sum_{n=1}^{\infty}\left(f_n\left(\Omega,t\right)e^{in\varphi}+c.c.\right)$$

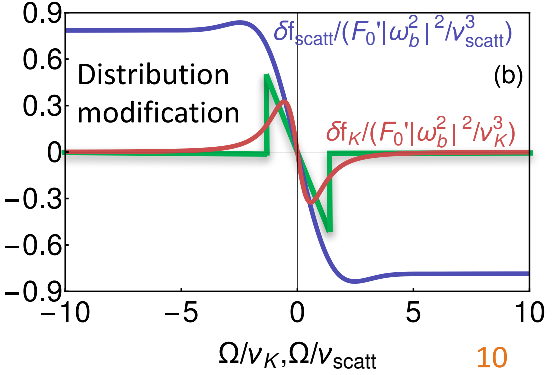
Near marginal stability, a perturbation theory can be developed in orders of $\omega_b^2/\nu_{K,scatt}^2$ which leads to the ordering $\left|F_0'\right|\gg\left|f_1'^{(1)}\right|\gg\left|f_0'^{(2)}\right|,\left|f_2'^{(2)}\right|$. When memory effects are weak, i.e., $\nu_{K,scatt}/\left(\gamma_{L,0}-\gamma_d\right)\gg 1$,

$$f_1=\frac{\omega_b^2 F_0'}{2\left(i\Omega+\nu_K\right)}\qquad\qquad \frac{\partial f_0}{\partial t}+\frac{1}{2}\left(\omega_b^2\left[f_1'\right]^*+\omega_b^{2*}f_1'\right)=-\nu_K f_0$$

First-principles analytical determination of the collisional resonance broadening – part II

Blue curve: pitch-angle scattering
Red curve: Krook collisions
Green curve: previous heuristic broadening (Berk, NF '95)





When decoherence is strong, the distribution function has no angle dependence:

$$f(\Omega, t) \equiv F_0(\Omega) + f_0(\Omega, t)$$

In the limit $\nu_{K,scatt}/\left(\gamma_{L,0} - \gamma_d\right) \gg 1$, the distribution satisfies a diffusion equation:

$$\frac{\partial f(\Omega, t)}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial \Omega}\left[\left|\omega_b^2\right|^2 \mathcal{R}(\Omega)\frac{\partial f(\Omega, t)}{\partial \Omega}\right] = C\left[f, F_0\right]$$

With the spontaneously emerged collisional resonance functions (both satisfy $\int_{-\infty}^{\infty}\mathcal{R}(\Omega)d\Omega = 1$):

$$\mathcal{R}_K(\Omega) = \frac{1}{\pi\nu_K\left(1 + \Omega^2/\nu_K^2\right)} \qquad \mathcal{R}_{scatt}(\Omega) = \frac{1}{\pi\nu_{scatt}}\int_0^{\infty} ds\, \cos\left(\frac{\Omega s}{\nu_{scatt}}\right) e^{-s^3/3}$$

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Self-consistent formulation of collisional quasilinear transport theory near threshold

$$\frac{\partial f(\Omega,t)}{\partial t}-\frac{\pi}{2}\frac{\partial}{\partial\Omega}\left[\left|\omega_b^2\right|^2\mathcal{R}(\Omega)\frac{\partial f(\Omega,t)}{\partial\Omega}\right]=C\left[f,F_0\right]$$

$$\gamma_L(t)=\frac{\pi}{4}\int_{-\infty}^{\infty}d\Omega\mathcal{R}\frac{\partial f(\Omega,t)}{\partial\Omega}\qquad d\left|\omega_b^2\right|^2/dt=2\left(\gamma_L(t)-\gamma_d\right)\left|\omega_b^2\right|^2$$

- A QL theory naturally emerges when considering kinetic theory near threshold when collisions occur at a time scale faster than the phase mixing time scale.

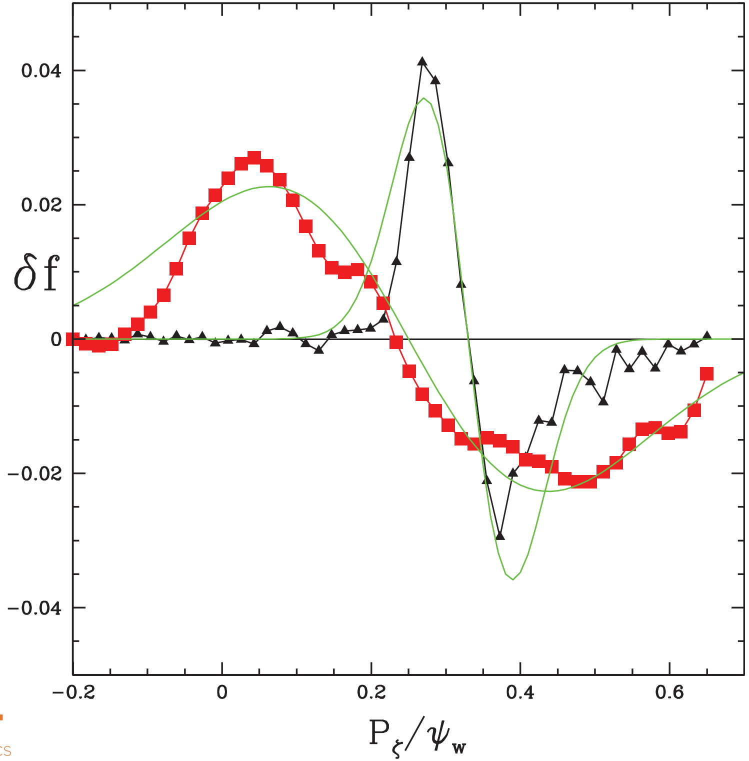
- The QL plasma system automatically replicates the nonlinear growth rate and the wave saturation levels $|\omega_{b,sat}|=8^{1/4}\left(1-\gamma_d/\gamma_{L,0}\right)^{1/4}\nu_K$ calculated from full kinetic theory near marginality,

$$\frac{d}{dt}\omega_B^2=(\gamma_L-\gamma_d)\omega_B^2(t)-\frac{\gamma_L}{2}\int_{t/2}^{t}dt'\,(t-t')^2\omega_B^2(t')\int_{t-t'}^{t'}dt_1\,\exp[-\nu(2t-t'-t_1)]\,\omega_B^2(t_1)\omega_B^2(t'+t_1-t)$$

(Berk, Breizman and Pekker, *Phys. Rev. Lett.* 1996)

Verification of the analytical predictions against ORBIT simulations of Alfvénic resonances

Modification of the distribution function vs canonical toroidal momentum



Red and black: guiding-center ORBIT simulation results for two different levels of collisionality

Green: analytic fit

White, Duarte *et al*, *Phys. Plasmas* **26**, 032508 (2019)

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The Resonance-broadened quasilinear (RBQ) code: a reduced, yet realistic approach to fast ion transport

[Gorelenkov, Duarte, Podestà and Berk, NF 2018, Duarte PhD thesis, 2017]

Workflow:

-background plasma profiles read from the TRANSP code

-eigenstructure calculated by the NOVA code

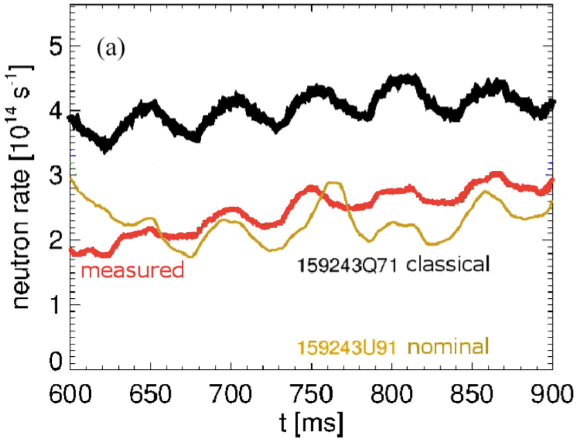
-damping rates and multi-dimensional resonance structure calculated by the NOVA-K code

-RBQ evolves the distribution function together with the amplitudes of the modes

Diffusion equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial I}\left(\sum_{n_k,p,m,m'} D(I;t)\right)\frac{\partial f}{\partial I} + \left(\left|\frac{\partial \Omega_{\mathbf{l}}}{\partial I}\right|_{I_r}\right)^{-2} \nu^3_{scatt,\mathbf{l}} \frac{\partial^2 (f - f_0)}{\partial I^2}$$

$$D(I;t) = \pi C_k^2(t)\,\mathcal{E}^2 \frac{\mathcal{R}(I - I_r)}{\left|\frac{\partial \Omega_{\mathbf{l}}}{\partial I}\right|} G^*_{m'p} G_{mp} \qquad \frac{\partial}{\partial I} = \omega \frac{\partial}{\partial \mathcal{E}} - n \frac{\partial}{\partial P_\varphi}$$



Mode amplitude evolution:

$$\frac{dC_n^2(t)}{dt} = 2\left(\gamma_{L,n} - \gamma_{d,n}\right) C_n^2(t)$$

Summary

- A systematic QL theory has been derived from first principles near an instability threshold, where the collisional resonance broadening functions emerge spontaneously
- The derivation indicates that QL theory can be applicable to a single discrete resonance (with no overlap), provided that stochasticity is large enough, as well as the usual overlapping regime
- An arbitrariness of collisional QL modeling (the shape of the resonance functions) has been removed
- The QL system (with the calculated broadening functions) systematically recovers the mode saturation levels for near-threshold plasmas previously calculated from nonlinear kinetic theory
- Resonance functions have been implemented into the Resonance Broadening Quasilinear (RBQ) code

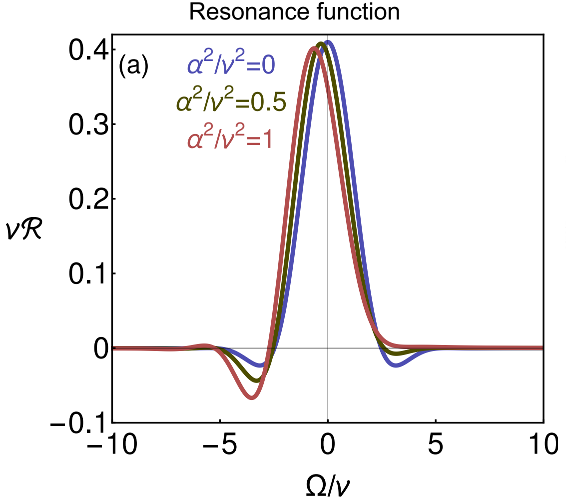
The use of the obtained resonance functions implies that fundamental features of nonlinear theory are automatically built into broadened QL theory

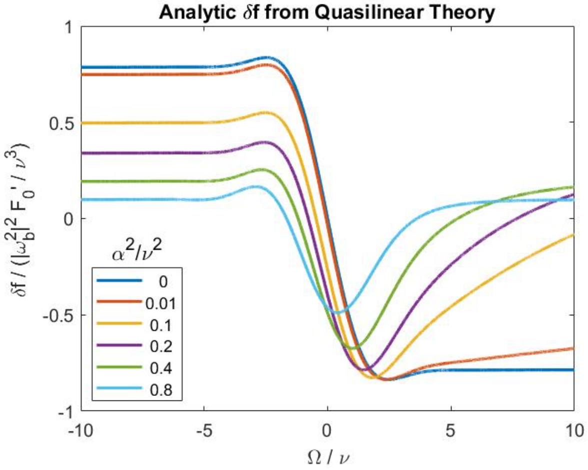
Duarte, Gorelenkov, White & Berk, "*Collisional resonance function in discrete-resonance quasilinear plasma systems*", *Phys. Plasmas* **26**, 120701 (December, 2019)

Ongoing work: extension of the model to account for collisional slowing down (drag)

$$\frac{\partial f}{\partial t}+\Omega\frac{\partial f}{\partial \varphi}+Re\left(\omega_b^2 e^{i\varphi}\right)\frac{\partial f}{\partial \Omega}=\nu^3\frac{\partial^2\left(f-F_0\right)}{\partial \Omega^2}+\alpha^2\frac{\partial\left(f-F_0\right)}{\partial \Omega}$$

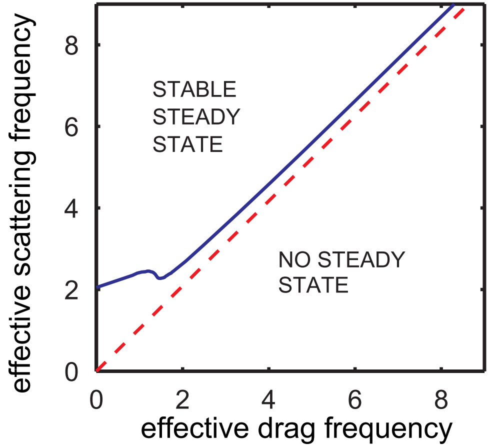
The main effect of drag is to introduce asymmetry and shift of the resonance





Comparison of the analytic results with BOT and ORBIT are under way!

Next step: to compare the quasilinear prediction of steady state response with the nonlinear prediction below:



(Lilley, Breizman & Sharapov, PRL 2009)

For the future: work oriented along the lines of the SciDAC ISEP project

- 2D implementation in RBQ
- More validation exercises
- Verification of new physics with ORBIT: saturation levels, timescale for mode evolution, broadening…
- Inclusion of zonal flows?

Backup slides

Verification: analytical collisional mode evolution near threshold

- Near marginal stability, the wave amplitude evolution is governed by [Berk, Breizman and Pekker, PRL 1996]

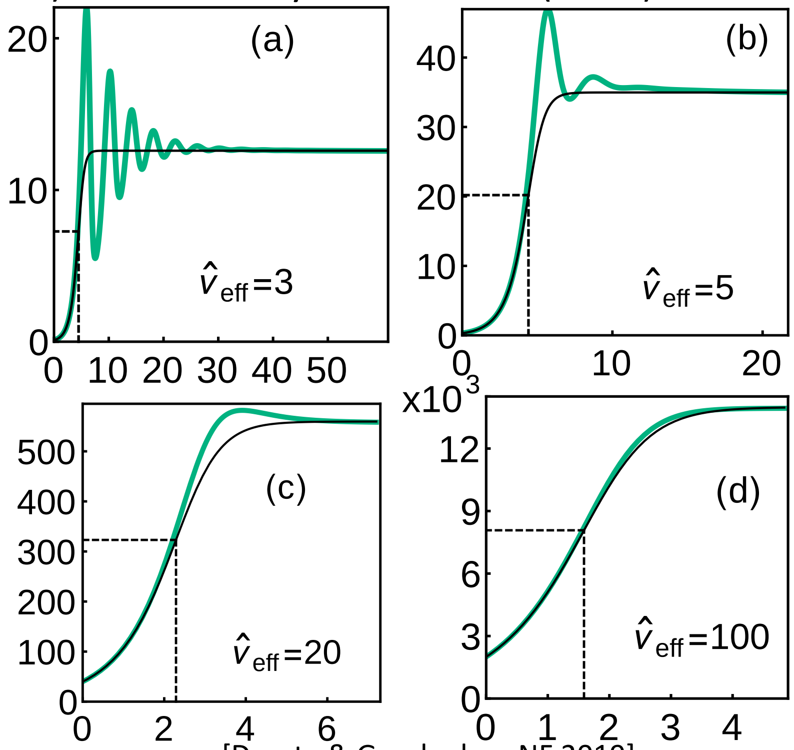
$$\frac{dA(t)}{dt} = A(t) - \frac{1}{2}\int d\Gamma\mathcal{H}\left\{\int_0^{t/2} dz z^2 A(t-z)\times\right.$$
$$\left.\times\int_0^{t-2z} dy e^{-\hat{\nu}_{eff}^3 z^2(2z/3+y)} A(t-z-y)A^*(t-2z-y)\right\}$$

- An approximate analytical solution is found when $\hat{\nu}_{eff} \gg 1$: [Duarte & Gorelenkov, NF 2019]

$$A(t) = \frac{A(0)e^t}{\sqrt{1-gA^2(0)\left(1-e^{2t}\right)}}$$

$g \equiv \int d\Gamma\mathcal{H}\frac{\Gamma(1/3)}{6\hat{\nu}_{eff}^4}\left(\frac{3}{2}\right)^{1/3}$ is a resonance-averaged collisional contribution evaluated by NOVA-K

Amplitude A vs time t for the full cubic equation (green) and the analytical solution (black)



[Duarte & Gorelenkov, NF 2019]

- same form of the function calculated by Dupree [T. H. Dupree, Phys. Fluids **9**, 1773 (1966)] in a different context, namely in the study of strong turbulence theory, where a dense spectrum of fluctuations diffuse particles away from their free-streaming trajectories. In that case, the cubic term in the argument of the exponential is proportional to a collisionless diffusion coefficient.
- the reduction of reversible equations of motion into a diffusive system of equations that governs the resonant particle dynamics without detailed tracking of the ballistic motion
- The collisional broadening of resonance lines is a universal phenomenon in physics (e.g., atoms emission/absorption spectral profile in atomic physics)